Stable semigroups on homogeneous trees and hyperbolic spaces

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Abstract

We prove the kernel estimates related to subordinated semigroups on homogeneous trees. We study the long time propagation problem. We exploit this to show exit time estimates for (large) balls. We use an abstract setting of metric measure spaces. This enables us to give these results for trees end hyperbolic spaces as well. Finally, we show some estimates for the Poisson kernel of a ball.

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1 Introduction

In 1961 Getoor [16] proposed subordinated semigroups in the context of the real hyperbolic spaces. It is only recently when the corresponding kernel estimates were found ([1], [18]).

The aim of this paper is to give a corresponding result in the context of homogeneous trees. Our motivations come from the fact that such structures make a discrete setting counterpart for hyperbolic spaces. Large scale analogy holds not only in geometry but also in analysis, see e.g. [12], [13], [8].

Our starting point is a diffusion semigroup considered in [8]. By subordination we obtain a new semigroup, which is referred to as to the stable one. We show estimates for the corresponding kernel (Theorem 3.1 below). In the proof we use time-space relations discovered in [18]. On the other hand, our theorem leads to a natural interpretation for the analogous result from [18] (see remarks after the proof).

Next, we consider the long time propagation problem (Theorem 3.2). It turns out that for large time t the mass of our kernel is distributed at distances comparable with $t^{2/\alpha}$. We give two different proofs. First of them is of general nature and exploits some properties of the underlying diffusion semigroup. This works for hyperbolic spaces or Riemannian manifolds as well. The other proof, a very simple one, shows that our Theorem 3.1 is useful as well.

Getoor [16] raised the question of "stability" properties for semigroups of this type. Obviously, here we have neither classical scaling, nor its *weak* form which is typical for e.g. fractals [7]. However, one may interpret Theorem 3.2 as an *asymptotical* scaling property. A sample of its consequences is given in the last section.

We conclude the paper by giving an application of Theorem 3.1. We study exit time from balls for the *stable* process corresponding to our semigroup. For related results we refer the reader to [17] or [20]. In general, we were inspired by the approach from [5], for stable case see [7]. The results in section 4 have their analogues in these papers. Observe, however, that the argument of [5] and [7] hinges on the *Ahlfors-regularity* of the measure, i.e. polynomial volume growth. Clearly, this excludes the homogeneous trees and hyperbolic spaces. Our contribution is to make it available for stable processes in exponential volume growth setting. Moreover, we give a proof in an abstract framework of metric measures spaces (cf. [14]). The interplay between (21) and (22) below may be of independent interest. In this way, we get our results for homogeneous trees and hyperbolic spaces at the same time.

Finally, using the Ikeda-Watanabe formula we give estimates for the Poisson kernel for balls.

2 Preliminaries

Consider the nearest-neighbor Laplacian Δ and the related heat semigroup $\mathcal{H}_t$ with continuous time on a homogeneous tree X of degree $q+1$ with $q \geq 2$, i.e.

$$\Delta f(x) = f(x) - \frac{1}{q+1} \sum_{y \sim x} f(y), \quad x \in X \quad \text{and} \quad \mathcal{H}_t = e^{-t\Delta}, \quad t > 0.$$

See [8] for detailed exposition. We adopt the general setting from this paper. For the reader's convenience we recall definitions needed in what follows. In particular, let h_t denote the corresponding heat kernel and $h_t^{\mathbb{Z}}$ the heat kernel in the one-dimensional case. Moreover, set $\gamma = \frac{2\sqrt{q}}{q+1}$ so that $b_2 = 1 - \gamma$ is the bottom of the spectrum of the Laplacian acting on $L^2(X)$.

We adopt the convention that c (without subscripts) denotes a generic constant whose value may change from one place to another. To avoid some curiosities occasionally we write $\tilde{c}$, c'... with the same properties. Numbered constants (with subscripts) always keep their particular value throughout the current theorem or proof. We often write $f \asymp g$ to indicate that there exists $c > 0$ such that $c^{-1} < f/g < c$. Similarly, $f(x) \asymp g(x)$, $x \to \infty$, means $f \asymp g$ for x large enough.

The kernel h_t is known to satisfy the following estimates [8]:

$$h_t(x) \asymp \frac{e^{-b_2 t}}{t} \phi_0(x) h_{t\gamma}^{\mathbb{Z}}(|x|+1),$$

where

$$\phi_0(x) = \left(1 + \frac{q-1}{q+1}|x|\right) q^{-\frac{|x|}{2}}, \quad x \in X \tag{1}$$

is the *spherical function*,

$$h_t^{\mathbb{Z}}(j) = e^{-t} I_{|j|}(t), \quad t > 0, \, j \in \mathbb{Z},$$

and $I_\nu(t)$ stands for the modified Bessel function of the first kind. Consequently,

$$h_t(x) \asymp \frac{e^{-t}}{t}\phi_0(x)I_{1+|x|}(t\gamma), \quad t>0,\ x\in X. \tag{2}$$

In what follows we fix $\alpha\in(0,2)$ and consider the *subordinate semigroup*

$$e^{-t\Delta^{\alpha/2}} = \int_0^\infty e^{-u\Delta}\eta_t(u)du,$$

where the *subordinator* $\eta_t(\cdot)$ is a function (defined on $\mathbb{R}^+$) determined by its Laplace transform,

$$\mathcal{L}[\eta_t(\cdot)](\lambda) = e^{-t\lambda^{\alpha/2}}.$$

For the corresponding kernels we have

$$p_t(x) = \int_0^\infty h_u(x)\eta_t(u)du. \tag{3}$$

Sometimes we refer to $p_t(x)$ as to the α-stable kernel. For more details concerning this construction we refer the reader e.g. to [6].

3 α-stable kernel

The main result may be stated as follows.

Theorem 3.1. *For any constants $K, M>0$*

$$p_t(x) \asymp \begin{cases} \phi_0(x)t^{-3/2}\exp(-t(1-\gamma)^{\alpha/2}), & |x|<Kt^{1/2},\ t\geq 1,\\ \phi_0(x)t|x|^{-2-\alpha/2}q^{-|x|/2}, & |x|>Mt^{2/\alpha}>0. \end{cases} \tag{4}$$

Proof. First, we collect some auxiliary estimates for Bessel function $I_\nu(z)$. Recall its integral representation (e.g. [15], (8.431.1))

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}}\int_{-1}^1 (1-u^2)^{\nu-1/2}e^{-zu}du = \frac{(2\pi z)^{-1/2}e^z}{2^{\nu-1/2}\Gamma(\nu+1/2)}\int_0^{2z}[u(2-u/z)]^{\nu-1/2}e^{-u}du.$$

Clearly, the last integral is bounded above by $2^{\nu-1/2}\Gamma(\nu+1/2)$ so that

$$I_\nu(z)\leq cz^{-1/2}e^z, \quad z>0,\ \nu>0. \tag{5}$$

Let us recall that ([8])

$$I_\nu(z) \asymp \frac{e^{\sqrt{\nu^2+z^2}}}{\sqrt{z+\nu}}\left(\frac{z}{\nu+\sqrt{\nu^2+z^2}}\right)^\nu, \quad \nu\geq 1,\ z>0. \tag{6}$$

Assume that $z>\max(1,a\nu^2)$ with some $a\in(0,1)$ and $\nu>1$. Thus,$\sqrt{\nu^2+z^2}-z\leq a/2$ so that $\exp(\sqrt{\nu^2+z^2})\asymp\exp(z)$ (in the lower bound there is a constant that depends on a). Clearly, $\sqrt{z+v}\asymp\sqrt{z}$ and the quotient in the parentheses in (6) is bounded above by 1. Moreover,

$$\frac{z^\nu}{(\nu+\sqrt{\nu^2+z^2})^\nu} \geq \frac{1}{(\sqrt{a}/\sqrt{z}+\sqrt{1+a/z})^{\sqrt{az}}} \geq \frac{1}{(1+2\sqrt{a}/\sqrt{z})^{\sqrt{z}/(2\sqrt{a})\times 2a}} \geq \frac{1}{e^{2a}}.$$

Consequently, we obtain the desired simplification

$$I_\nu(z)\asymp z^{-1/2}e^z, \quad z>\max(1,a\nu^2),\ \nu\geq 1. \tag{7}$$

We recall the exact estimates of the densities $\eta_t(\cdot)$ which will be fundamental in what follows (see e.g. [18]). We have

$$\eta_t(u)\asymp t^{\frac{1}{2-\alpha}}u^{-\frac{4-\alpha}{4-2\alpha}}\exp\left(-c_1t^{\frac{2}{2-\alpha}}u^{-\frac{\alpha}{2-\alpha}}\right), \qquad t^{-2/\alpha}u<c, \tag{8}$$

where $c_1=c_1(\alpha)=\dfrac{2-\alpha}{2}\left(\dfrac{\alpha}{2}\right)^{\frac{\alpha}{2-\alpha}}$ and

$$\eta_t(u)\asymp tu^{-1-\alpha/2}, \qquad t^{-2/\alpha}u>c. \tag{9}$$

According to (8) and (9), it is convenient to split the integral (2) as follows

$$\begin{aligned}
p_t(x) &= \int_0^{c_0 t^{2/\alpha}} h_u(x)\eta_t(u)du + \int_{c_0 t^{2/\alpha}}^{\infty} h_u(x)\eta_t(u)du \\
&= \phi_0(x) t^{\frac{1}{2-\alpha}} \int_0^{c_0 t^{2/\alpha}} e^{-u} I_{1+|x|}(\gamma u) u^{-\frac{4-\alpha}{4-2\alpha}-1} \exp(-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}) du \\
&+ \phi_0(x) t \int_{c_0 t^{2/\alpha}}^{\infty} e^{-u} I_{1+|x|}(\gamma u) u^{-2-\alpha/2} du \qquad (10) \\
&= \phi_0(x) \left(\mathbf{A}^{(x,t)} + \mathbf{B}^{(x,t)} \right).
\end{aligned}$$

Now, we assume that $c_0 = 1$ and $|x| \le K\sqrt{t}$ with x and t large enough. Note that neither x, nor t is fixed. It follows that $(1+|x|)^2 \le (1+K\sqrt{t})^2 \le \gamma t^{2/\alpha}$. Hence, by (7) with $a = 1$ we get

$$I_{1+|x|}(\gamma u) \le c u^{-1/2} e^{\gamma u}, \quad u > t^{2/\alpha}.$$

In consequence,

$$\begin{aligned}
\mathbf{B}^{(x,t)} &\le ct \int_{t^{2/\alpha}}^{\infty} e^{-(1-\gamma)u} u^{-(5+\alpha)/2} du \\
&\le ct^{-5/\alpha} \int_{t^{2/\alpha}}^{\infty} e^{-(1-\gamma)u} du \\
&= ct^{-5/\alpha} e^{-(1-\gamma)t^{2/\alpha}}
\end{aligned}$$

To estimate $\mathbf{A}^{(x,t)}$ let us split it as follows

$$\mathbf{A}^{(x,t)} = t^{\frac{1}{2-\alpha}} \left(\int_0^{\alpha t/2} + \int_{\alpha t/2}^{t^{2/\alpha}} \right) e^{-u} I_{1+|x|}(\gamma u) u^{-\frac{4-\alpha}{4-2\alpha}-1} \exp(-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}) du = \mathbf{A}_1^{(x,t)} + \mathbf{A}_2^{(x,t)}.$$

Now, apply (7) to the integral $\mathbf{A}_2^{(x,t)}$. After simple change of the variable $u \to tu$, we get

$$\mathbf{A}_2^{(x,t)} \asymp ct^{-1} \int_{\frac{\alpha}{2}}^{t^{\frac{2}{\alpha}-1}} u^{-\frac{4-\alpha}{4-2\alpha}-\frac{3}{2}} \exp(-t((1-\gamma)u + c_1 u^{-\frac{\alpha}{2-\alpha}})) du.$$

Observe that the minimum of function $p(u) = (1-\gamma)u + c_1 u^{-\frac{\alpha}{2-\alpha}}$ is attained at

$$u_0 = \frac{\left(\frac{\alpha c_1}{2-\alpha}\right)^{1-\frac{\alpha}{2}}}{(1-\gamma)^{1-\alpha/2}}.$$

Since

$$\left(\frac{\alpha c_1}{2-\alpha}\right)^{1-\frac{\alpha}{2}} = \left[\frac{\alpha}{2-\alpha}\frac{2-\alpha}{2}\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2-\alpha}}\right]^{\frac{2-\alpha}{2}} = \left[\left(\frac{\alpha}{2}\right)^{\frac{2-\alpha}{2}}\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2}}\right] = \frac{\alpha}{2},$$

we get

$$u_0 = \frac{\alpha/2}{(1-\gamma)^{1-\alpha/2}}.$$

Hence, for t large enough u_0 is in the integration range and $p(u_0) = (1-\gamma)^{\alpha/2}$. Obviously, our integral is bounded by integrals with limits fixed

$$\int_{\frac{\alpha}{2}}^{u_0} \le \int_{\frac{\alpha}{2}}^{t^{\frac{2}{\alpha}-1}} \le \int_0^{\infty}.$$

The Laplace method [22] applied to the extreme members of this inequality gives the same result, so that we obtain the asymptotic of our integral:

$$ct^{-1/2} e^{-tp(u_0)}, \quad t \to \infty.$$

Consequently,

$$\mathbf{A}_2^{(x,t)} \asymp t^{-3/2} \exp(-(1-\gamma)^{\alpha/2} t), \quad |x| < K\sqrt{t}$$

and $t \geq 1$, say. Similarly, using (5) we get

$$\mathbf{A}_1^{(x,t)} \leq ct^{-1} \int_{\frac{\alpha}{2}}^{t^{\frac{2}{\alpha}-1}} u^{-\frac{4-\alpha}{4-2\alpha}-\frac{3}{2}} \exp(-t((1-\gamma)u + c_1 u^{-\frac{\alpha}{2-\alpha}}))du.$$

Since the minimum of $p(u)$ is *not* attained in $(0, \alpha/2)$, in this case the Laplace method gives the following lower bound:

$$\mathbf{A}_1^{(x,t)} \leq ct^{-2} \exp(-p(\alpha/2)t).$$

It follows that $p_t(x) \asymp \mathbf{A}_2^{(x,t)}$ and the first of the desired estimates follows.

Now, assume that $|x| > Mt^{2/\alpha}$. Since we consider large $|x|$ only (or even $|x| \to \infty$), we may and do put $|x| - 1$ in place of $|x|$ when estimating $p_t(\cdot)$. This simplifies the notation. We put $c_0 = aM$ in the decomposition (10), where $a \in (0,1)$ is to be specified later. Then, by (6) and the elementary inequalities $e^{\sqrt{|x|^2+\gamma^2u^2}} \leq e^{|x|+\gamma u}$, $|x| + \sqrt{|x|^2+\gamma^2u^2} \geq 2|x|$, we get

$$\begin{aligned}
\mathbf{A}^{(x,t)} &= t^{\frac{1}{2-\alpha}} \int_0^{aMt^{2/\alpha}} e^{-u} I_{|x|}(\gamma u) u^{-\frac{4-\alpha}{4-2\alpha}-1} e^{-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}} du \\
&\leq c|x|^{\frac{\alpha}{4-2\alpha}} \int_0^{a|x|} \frac{e^{\sqrt{|x|^2+\gamma^2u^2}-u}(\gamma u)^{|x|} u^{-\frac{4-\alpha}{4-2\alpha}-1} e^{-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}}}{\sqrt{|x|+\gamma u}\left(|x| + \sqrt{|x|^2+\gamma^2u^2}\right)^{|x|}} du \\
&\leq c|x|^{\frac{\alpha}{4-2\alpha}-\frac{1}{2}} \left(\frac{ae\gamma}{2}\right)^{|x|} \int_0^{a|x|} e^{-(1-\gamma)u} u^{-\frac{4-\alpha}{4-2\alpha}-1} e^{-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}} du.
\end{aligned}$$

Clearly, the last integral is convergent and bounded above by a constant independent of $|x|$. Therefore,

$$\mathbf{A}^{(x,t)} \leq c|x|^{\frac{\alpha}{4-2\alpha}-\frac{1}{2}} \left(\frac{ae\gamma}{2}\right)^{|x|}. \tag{11}$$

On the other hand, again by (6) and the change of variable $u \to ux$, we obtain

$$\begin{aligned}
\mathbf{B}^{(x,t)} &= t \int_{aMt^{2/\alpha}}^{\infty} I_{|x|}(\gamma u) u^{-2-\alpha/2} e^{-u} du \\
&\geq ct \int_{a|x|}^{\infty} \frac{e^{\sqrt{|x|^2+\gamma^2u^2}-u}}{\sqrt{|x|+\gamma u}} \frac{(\gamma u)^{|x|} u^{-2-\alpha/2}}{\left(|x| + \sqrt{|x|^2+\gamma^2u^2}\right)^{|x|}} du \\
&\geq ct\gamma^{|x|}|x|^{-\frac{\alpha+3}{2}} \int_a^{\infty} \frac{e^{|x|(\sqrt{1+\gamma^2u^2}-u)} u^{-2-\alpha/2+|x|}}{\sqrt{1+\gamma u}\left(1 + \sqrt{1+\gamma^2u^2}\right)^{|x|}} du \\
&\asymp t\gamma^{|x|}|x|^{-\frac{\alpha+3}{2}} \int_a^{\infty} e^{|x|\left(\sqrt{1+\gamma^2u^2}-u+\log(u)-\log(1+\sqrt{1+\gamma^2u^2})\right)} \frac{u^{-2-\alpha/2}}{\sqrt{1+\gamma u}} du.
\end{aligned}$$

Observe that the same calculation with the lower limit of integration equal to 0 gives the opposite bound. Let

$$p(u) = \sqrt{1+\gamma^2u^2} - u + \log(u) - \log(1 + \sqrt{1+\gamma^2u^2})$$

and $g = \sqrt{1+\gamma^2u^2}$. Then $p'(u) = -1 + g/u$ and, consequently, $p(u)$ attains the maximum at $u_0 = \frac{q+1}{q-1} > 1$. Hence, u_0 belongs to the integration range for integrals in both upper and lower bound for $\mathbf{B}^{(x,t)}$. Consequently, by the Laplace method, both of them have the same asymptotic as $|x| \to \infty$. Since

$$p(u_0) = \sqrt{1 + \frac{4q}{(q-1)^2}} - \frac{q+1}{q-1} + \log\left(\frac{q+1}{(q-1)\left(1+\sqrt{1+\frac{4q}{(q-1)^2}}\right)}\right) = -\log\left(\frac{2q}{q+1}\right) = -\log(\gamma\sqrt{q}),$$

it follows that

$$\mathbf{B}^{(x,t)} \asymp t|x|^{-2-\alpha/2} e^{|x|(\log\gamma - \log(\gamma\sqrt{q}))} = t|x|^{-2-\alpha/2} q^{-|x|/2}, \quad |x| \geq Mt^{2/\alpha}$$

and $|x|$ is large enough (and hence for $|x| > 1$). Moreover, if we take $a = 1/e$ then $ae\gamma/2 \leq q^{-1/2}$ so that $\mathbf{A}^{(x,t)} = o(\mathbf{B}^{(x,t)})$, $|x| \to \infty$ and $p_t(x) \asymp \mathbf{B}^{(x,t)}$. The assertion follows. $\square$

Remark 1. Our theorem can be compared with the following result of [18]. For reader's convenience we give it below, specialized to the (real) hyperbolic space $\mathbb{H}^n$. The corresponding α-stable kernel and spherical function are denoted with the tilde.

Theorem. [[18], Corollary 5.6] Let $|\rho| = (n-1)/2$. For any constants $K, M > 0$ and $t + |x| > 1$ we have

$$\tilde{p}_t(x) \asymp \begin{cases} \tilde{\phi}_0(x)\, t^{-3/2}\, e^{-|\rho|^\alpha t}, & |x| \le K\, t^{1/2} \\ \tilde{\phi}_0(x)\, t|x|^{-2-\alpha/2} e^{-|\rho||x|}, & |x| \ge M\, t^{2/\alpha}. \end{cases} \tag{12}$$

In the context of hyperbolic space (or, more generally, symmetric space of non-compact type), the parameter $|\rho|$ plays a double role: it is the square root of the bottom of the spectrum of the Laplace-Beltrami operator; at the same time, the volume growth of the ball of the radius r is equivalent to $e^{2|\rho|r}$, $r \to \infty$. One may ask, whether it is the spectral data or the geometry which appears in the above estimates. The comparison with Theorem 3.1 gives us a natural interpretation: in the first part (i.e. in the long time asymptotics) we deal with the spectral data, in the other case the volume growth intervenes.

Remark 2. Note that for the remaining region $Kt^{1/2} < |x| < Mt^{2/\alpha}$, in the continuous setting there is no *simple explicit* estimate of $\tilde{p}_t(x)$ (see [18], Corollary 5.6).

The Brownian motion and α-stable processes in $\mathbb{R}^d$ share the same type of long time heat repartition. Namely, with the standard understanding that $\alpha = 2$ corresponds to the Brownian motion, for $A_1 < A_2$ we have

$$\int_{A_1 t^{1/\alpha} \le |x| \le A_2 t^{1/\alpha}} p_t(x)dx = c(A_1, A_2) \in (0,1).$$

This follows immediately from the scaling property

$$p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x). \tag{13}$$

Moreover, $c(A_1, A_2) \to 1$ if $A_1 \to 0$ and $A_2 \to \infty$ so that

$$\int_{A_1 t^\beta \le |x| \le A_2 t^\beta} p_t(x)dx \to 0, \quad t \to \infty, \tag{14}$$

provided $\beta \neq 1/\alpha$ (cf. [2], p. 50).

On the other hand, for the Brownian motion in the (real) hyperbolic space $\mathbb{H}^n$, a non-classical phenomenon of concentration was observed in [10]. Namely,

$$\int_{A_1 t \le |x| \le A_2 t} h_t(x)dx \to 1, \quad t \to \infty,$$

provided $A_1 < n-1 < A_2$. The change of the space-time scaling should be noted. This result was sharpened and generalized to symmetric space setting ([2], [4]). In the context of homogeneous trees the analogous result was shown in [21] and [23]:

$$\sum_{R_0 t - r(t) \le |x| \le R_0 t + r(t)} h_t(x) \to 1, \quad t \to \infty,$$

where $R_0 = (q-1)/(q+1)$ and $r(t)$ is a positive function such that $r(t)t^{-1/2} \to \infty$, $t \to \infty$. This might suggest a hypothesis of the same kind for our kernel $p_t(x)$, e.g.

$$\sum_{A_1 t^{2/\alpha} \le |x| \le A_2 t^{2/\alpha}} p_t(x) \to 1, \quad t \to \infty.$$

The following theorem shows that this is not the case.

Theorem 3.2. *For $0 < A_1 < A_2$ let $R(t) = \{(x,t) \in X \times \mathbb{R}^+ : A_1 t^{2/\alpha} \le |x| \le A_2 t^{2/\alpha}\}$. Then there exist c_1 and c_2 such that*

$$0 < c_1 < \sum_{x \in R(t)} p_t(x) < c_2 < 1, \quad t \to \infty. \tag{15}$$

Conversely, for any given $0 < c_1 < 1$ ($0 < c_2 < 1$ resp.) there exist A_1 and A_2 such that (15) holds true with some c_2 (c_1 resp.).

Proof. Set $R_0 = (q-1)/(q+1)$ and let R_1, R_2 be such that $R_1 < R_0 < R_2$. Then, by Theorem 1 of [21], we have

$$\sum_{R_1 u \le |x| \le R_2 u} h_u(x) \to 1, \qquad u \to \infty. \tag{16}$$

Moreover, let $c_3 = A_1/R_1$ and $c_4 = A_2/R_2$. We require additionally that R_1 and R_2 be close to R_0 so that $c_3 < c_4$. Then $c_3 t^{2/\alpha} < u < c_4 t^{2/\alpha}$ yields

$$|x| \in (R_1 u, R_2 u) \implies x \in R(t). \tag{17}$$

From the definition of $p_t(x)$, (16) and (17), we get

$$\begin{aligned}
\sum_{x \in R(t)} p_t(x) &= \int_0^\infty \left(\sum_{x \in R(t)} h_u(x) \right) \eta_t(u) du \\
&\ge \int_{c_3 t^{2/\alpha}}^{c_4 t^{2/\alpha}} \left(\sum_{R_1 u \le |x| \le R_2 u} h_u(x) \right) \eta_t(u) du \\
&\to \int_{c_3 t^{2/\alpha}}^{c_4 t^{2/\alpha}} \eta_t(u) du, \quad t \to \infty.
\end{aligned}$$

Formally, the last integral depends on t. By the scaling property (13), however, it evaluates to

$$t^{-2/\alpha} \int_{c_3 t^{2/\alpha}}^{c_4 t^{2/\alpha}} \eta_1(t^{-2/\alpha} u) du = \int_{c_3}^{c_4} \eta_1(u) du = c_0. \tag{18}$$

This is an absolute constant which depends on c_3, c_4 and α only. The lower bound in the first assertion follows. Since the lower bound is true for *any* $A_1 < A_2$, the mass of the annulus $R(t)$ (with A_1 and A_2 fixed) is strictly less than 1. In other words, $c_2 < 1$ in (15) and there is no *total* mass concentration observed. The proof of the first assertion is complete.

The second assertion follows from the fact that c_0 in (18) can be required to take any value in $(0,1)$. Indeed, if $A_1 \to 0$ and $A_2 \to \infty$ then we may fix $R_1 < R_0 < R_2$ independently of A_1 and A_2, so that $c_3 \to 0$ and $c_4 \to \infty$. Since $\int_0^\infty \eta_t(u) = 1$ we may require c_1 to be arbitrarily close to 1.

Further, fix any $0 < \tilde{c}_2$. The upper bound for the mass of the annulus $\tilde{R}(t) = \{(x,t) \in X \times \mathbb{R}^+ : \tilde{A}_1 t^{2/\alpha} \le |x| \le \tilde{A}_2 t^{2/\alpha}\}$ follows from the lower bound for $R(t)$ provided $A_2 < \tilde{A}_1$. Since the mass of $R(t)$ can be required to be arbitrarily close to 1, $t \to \infty$, the mass of $\tilde{R}(t)$ is (asymptotically) smaller than $\tilde{c}_2$. The proof is complete. □

The following corollary is an analogue of the classical counterpart (14).

Corollary 3.3. *For $0 < \tilde{A}_1 < \tilde{A}_2$ and some $\beta > 0$ let $\bar{R}(t) = \{(x,t) \in X \times \mathbb{R}^+ : \tilde{A}_1 t^\beta \le |x| \le \tilde{A}_2 t^\beta\}$. If $\beta \neq 2/\alpha$ then*

$$\sum_{x \in \bar{R}(t)} p_t(x) \longrightarrow 0, \quad t \to \infty.$$

Proof. For t large enough, $R(t)$ and $\bar{R}(t)$ are disjoint. □

Evidently, space-time scaling in (15) is characteristic for the Brownian motion in hyperbolic spaces and homogeneous trees. On the other hand, the concentration phenomenon is not observed. From the probabilistic point of view this may be explained by the influence of the long jumps of the corresponding stable process. Indeed, the *Lévy measure* is of the same exponential order as volume growth because it arises from the second estimate in (4). Actually, we have

Corollary 3.4. *Let $\nu(x) := \lim_{t\to 0} p_t(x)/t$ be the Lévy measure for our semigroup. Then*

$$\nu(x) \asymp |x|^{-1-\alpha/2} q^{-|x|}, \quad |x| \ge 1.$$

Proof. From Theorem 3.1 and (1) we get

$$\nu(x) \asymp \phi_0(x) |x|^{-2-\alpha/2} q^{-|x|/2} \asymp |x|^{-1-\alpha/2} q^{-|x|}.$$

□

Clearly, the proof of Theorem 3.2 with minor modifications only can be applied in the context of the symmetric spaces with Theorem 1 of [2] instead of (16). We prefer, however, to take the opportunity given by Theorem 2 of that article to state our result in the more general setting of manifolds. For reader's convenience, we recall the framework. We assume that M is a complete, noncompact Riemannian manifold with the volume growth controlled by

$$\mathrm{vol}(B(x,r)) = O(r^{\kappa} e^{2Kr}), \quad r \to \infty,$$

with some constants κ and K, and the spectral gap $E^2 = \inf \mathrm{spec}(-\Delta) > 0$. In general we have $E \leq K$, while in symmetric spaces $E = K = |\rho|$. Set $R_1 = 2(K - \sqrt{K^2 - E^2})$, $R_2 = 2(K + \sqrt{K^2 - E^2})$. Let $A(t)$ be a function such that

$$\begin{array}{ll} A(t) - \dfrac{\kappa - 1}{2\sqrt{K^2 - E^2}} \log t \nearrow \infty & \text{if} \quad K < E, \\ A(t) = (2\kappa t \log t)^{1/2} & \text{if} \quad K = E \ \text{ and } \ \kappa > 0, \\ A(t) t^{-1/2} \nearrow \infty & \text{if} \quad K = E \ \text{ and } \ \kappa = 0. \end{array}$$

Since the heat kernel depends on two variables (and is denoted by $h_t(x,y)$), we fix arbitrary $y \in M$ and redefine slighlty $R(t) = \{(x,t) \in X \times \mathbb{R}^+ : A_1 t^{2/\alpha} \leq d(x,y) \leq A_2 t^{2/\alpha}\}$. By Theorem 2 from [2]

$$\int_{R_1 t - A(t) \leq d(x,y) \leq R_2 t + A(t)} h_t(x,y) \to 1, \quad t \to \infty.$$

Note that in any case we may and do require $A(t) = o(t)$, which is essential for our proof to work (cf. (17)). Thus, we arrive at

Corollary 3.5. *For any constants $0 < A_1 < A_2$ there exist c_1 and c_2 such that*

$$0 < c_1 < \int_{R(t)} p_t(x) dx < c_2 < 1, \quad t \to \infty.$$

Changing A_1 and A_2 we may require c_1 to be close to 1 or c_2 to be close to 0.

Below we include an alternative approach that relies directly on the α-stable kernel estimates (4). It shows, in a sense, that the α-stable kernel mass covered by Theorem 3.1 is large enough to be useful in some applications.

Second proof of Theorem 3.2. For $x \in R(t)$ we have

$$p_t(x) \asymp t \phi_0(x) |x|^{-2-\alpha/2} q^{-|x|/2}. \tag{19}$$

By (1),

$$\phi_0(x) \asymp |x| q^{-|x|/2}, \quad |x| \to \infty.$$

Therefore,

$$\sum_{x \in R(t)} p_t(x) \asymp t \sum_{x \in R(t)} |x|^{-1-\alpha/2} q^{-|x|}.$$

Since the function under the consideration depends only on the distance, we use "polar coordinates". At each sphere $\{|x| = n\}$ we have exactly $(q+1)q^n$ vertices, so that

$$\sum_{x \in R(t)} p_t(x) \asymp t \sum_{A_1 t^{2/\alpha} \leq n \leq A_2 t^{2/\alpha}} n^{-1-\alpha/2}.$$

Since obviously

$$\int_{a-2}^{b+2} y^{-1-\alpha/2} dy \asymp \int_a^b y^{-1-\alpha/2} dy$$

when $a \to \infty$ and $a/b < c_0 < 1$, it follows that

$$\sum_{x \in R(t)} p_t(x) \asymp t \int_{A_1 t^{2/\alpha}}^{A_2 t^{2/\alpha}} y^{-1-\alpha/2} dy.$$

Clearly, the last integral behaves as

$$(A_1^{-\alpha/2} - A_2^{-\alpha/2})(\alpha/2)^{-1} t^{-1}, \quad t \to \infty. \tag{20}$$

The lower bound in (15) follows. To obtain inequality $c_2 < 1$ in the upper bound, it is enough to take A_1 sufficiently large. To allow any value of $A_1 < A_2$, we can repeat here the argument following (18) in the previous proof. The assertion follows. $\square$

Remark. Clearly, (19) holds also for $\bar{R}(t)$ with $\beta > 2/\alpha$ as well. In this case, (20) implies that

$$\sum_{x \in \bar{R}(t)} p_t(x) \to 0, \quad t \to \infty.$$

However, this direct argument fails for $\beta < 2/\alpha$. Actually, if (19) holded for $|x| \geq At^\beta$ with some $\beta < 2/\alpha$, then we would obtain $t^{-\alpha\beta/2}$ in (20). Consequently, the mass of the annulus goes to infinity, which is impossible (cf. also Proposition 5.4 of [18], where this was done for the line $|x| = t^\beta$ by an independent argument).

4 Exit time

We conclude our work by giving an application of Theorem 3.1. Since the results below are very similar for both homogeneous trees and hyperbolic spaces, we are tempted to use the following notation of metric spaces.

Let (E, μ) be a metric space with a measure μ that supports a heat kernel $p_t(x, y)$ in the sense of the axiomatic definition 2.1 of [14]. For the reader's convenience we recall it shortly. We assume that $p_t(\cdot, \cdot)$ is a $\mu \times \mu$ nonnegative measurable function and for μ-almost all $x, y \in E$ and all $s, t > 0$ we have $p_t(x, y) = p_t(y, x)$,

$$\int_E p_t(x, y) d\mu(y) = 1, \quad p_{t+s}(x, y) = \int_E p_t(x, z) p_s(z, y) d\mu(z),$$

and for each $u \in L^2(E, \mu)$

$$\int_E p_t(x, y) u(y) d\mu(y) \xrightarrow{L^2} u(x), \quad t \to 0^+.$$

In the case of the hyperbolic spaces or homogeneous trees we have $p_t(x, y) = p_t(d(x, y))$, where $d(x, y)$ is the distance. Under some general additional assumptions on X, this kernel gives rise to the associated Markov process (X_t, P_x), i.e.

$$P_x[X_t \in B] = \int_B p_t(x, y) d\mu(y).$$

For simplicity, we suppose that the space is homogeneous, i.e. there exists a function $V(r)$, called a volume growth, such that $V(r) = \mu(B(x, r))$, $x \in E$. It can be seen that for the proofs below this assumption is not essential and we could deal with non-homogeneous version $V(x, r)$ as well.

Moreover, assume that there exist $A \geq 1$ and $c_1 < 1$ such that

$$V(r) \leq c_1 V(r + A) \quad \text{and} \quad V(r + 1) \asymp V(r), \quad r \geq 1. \tag{21}$$

Actually, this covers the case of trees and hyperbolic spaces (with e.g. $A = 1$).

Furthermore, assume that for any $M > 0$

$$p_t(x, y) \asymp t d(x, y)^{-1-\alpha/2} V(d(x, y))^{-1}, \quad d(x, y) > Mt^{2/\alpha}, \quad d(x, y) > 1. \tag{22}$$

This is clearly satisfied in the context of trees and hyperbolic spaces as well (cf. Theorem 3.1 and (12) resp.).

Note that the first part of (21) implies that $\lim_{r \to \infty} V(r) = \infty$. In particular, our space is not contained in any ball. Below we use this fact without further mention.

Proposition 4.1. *For any $M > 0$ and $r > 1$ we have*

$$P_x[X_t \notin B(x, r)] \asymp t r^{-\alpha/2}, \quad r > Mt^{2/\alpha}.$$

Proof. By (22) we get

$$\begin{aligned} P_x[X_t \notin B(x, r)] &\asymp \int_{d(x,y)>r} p_t(x, y) d\mu(y) \\ &\asymp t \sum_{k=0}^{\infty} \int_{r+k<d(x,y)\leq r+k+1} d(x, y)^{-1-\alpha/2} V(d(x, y))^{-1} d\mu(y) \\ &\leq ct \sum_{k=0}^{\infty} (r + k)^{-1-\alpha/2} V(r + k)^{-1} (V(r + k + 1) - V(r + k)). \end{aligned}$$

Clearly, by (21) we get

$$V(r+k)^{-1}(V(r+k+1)-V(r+k)) \le c.$$

Moreover, by a comparison of the series with the corresponding integral it can be easily seen that

$$\sum_{k=0}^{\infty}(r+k)^{-1-\alpha/2} = r^{-1-\alpha/2} + \sum_{k=1}^{\infty}(r+k)^{-1-\alpha/2} \le r^{-\alpha/2} + \int_r^{\infty} z^{-1-\alpha/2}dz \le cr^{-\alpha/2}$$

and the upper bound in the assertion follows.

On the other hand we have similarly

$$\begin{aligned} P_x[X_t \notin B(x,r)] &\asymp t\sum_{k=0}^{\infty}\int_{r+kA<d(x,y)\le r+(k+1)A} d(x,y)^{-1-\alpha/2}V(d(x,y))^{-1}d\mu(y) \\ &\ge ct\sum_{k=0}^{\infty}(r+kA+A)^{-1-\alpha/2}\frac{V(r+kA+A)-V(r+kA)}{V(r+kA+A)}. \end{aligned}$$

Again, by (21)

$$\frac{V(r+kA+A)-V(r+kA)}{V(r+kA+A)} = 1 - \frac{V(r+kA)}{V(r+kA+A)} \ge 1-c_1 > 0.$$

Moreover,

$$\sum_{k=0}^{\infty}(r+kA+A)^{-1-\alpha/2} \ge \int_{r+A}^{\infty} z^{-1-\alpha/2}dz = c(r+A)^{-\alpha/2} \ge cr^{-\alpha/2},$$

since $r>1$. The proof is complete. □

For a measurable set D define the exit time $\tau_D = \inf\{t\ge 0;\ X_t \notin D\}$. Then

Proposition 4.2. *For any $M>0$ and $r>1$ we have*

$$P_x[\tau_{B(x,r)}<t] \le ctr^{-\alpha/2}, \quad r > Mt^{2/\alpha}.$$

Proof. The proof follows the lines of [5] (or [7]). Since it is short, we sketch it for the reader's convenience. Denote $T=\tau_{B(x,2r)}$. Then

$$\begin{aligned} P_x[T<t] &= P_x[X_t\notin B(x,r);\ T<t] + P_x[X_t \in B(x,r);\ T<t] \\ &\le P_x[X_t\notin B(x,r)] + P_x[X_t\in B(x,r);\ T<t] = A+B. \end{aligned}$$

By Proposition 4.1 we obtain $A \le ctr^{-\alpha/2}$. By the strong Markov property we have

$$\begin{aligned} B &= E_x[P_{X(T)}[X_{t-u}\in B(x,r)]_{|u=T};\ T<t] \\ &\le \sup_{u\le t}\sup_{z\in B(x,2r)^c} E_x[P_z[X_u\in B(x,r)];\ T<t] \\ &\le \sup_{u\le t}\sup_{z\in B(x,r)^c} E_x[P_z[X_u\notin B(z,r)];\ T<t] \\ &\le ctr^{-\alpha/2}. \end{aligned}$$

The proof is complete. □

Theorem 4.3. *For $r>1$*

$$E_y\tau_{B(x,r)} \le cr^{\alpha/2}, \quad y\in B(x,r)$$

and

$$E_x\tau_{B(x,r)} \asymp r^{\alpha/2}.$$

Proof. For any $y\in B(x,r)$ by Proposition 4.1 we have

$$P_y[\tau_{B(x,r)}>t] \le P_y[X_t\in B(x,r)] = 1-P_y[X_t\notin B(x,r)] \le 1-ctr^{-\alpha/2}$$

provided that $r>Mt^{2/\alpha}$ with some $M>0$. Let $t_0=r^{\alpha/2}$ so that for some c_0 we get

$$P_y[\tau_{B(x,r)}>t_0] \le 1-c_0. \tag{23}$$

Then, by Markov property, for $k = 1, 2, \ldots$ we have

$$\begin{aligned} P_y[\tau_{B(x,r)} > (k+1)t_0] &= P_y[\tau_{B(x,r)} \circ \theta_{t_0} > kt_0,\ \tau_{B(x,r)} > t_0] \\ &= E_y[P_{X(t_0)}[\tau_{B(x,r)} > kt_0];\ \tau_{B(x,r)} > t_0] \\ &\leq P_y[\tau_{B(x,r)} > t_0] \sup_{z \in B(x,r)} P_z[\tau_{B(x,r)} > kt_0] \end{aligned}$$

(here θ stands for the standard shift operator on the space of trajectories). By induction we get

$$P_y[\tau_{B(x,r)} > kt_0] \leq (1 - c_0)^k, \qquad y \in B(x,r), \quad k = 0, 1, 2, \ldots$$

Thus,

$$E_y \tau_{B(x,r)} = \int_0^\infty P_y[\tau_{B(x,r)} > t]dt \leq \sum_{k=0}^\infty t_0 P_y[\tau_{B(x,r)} > kt_0] \leq r^{\alpha/2} \sum_{k=0}^\infty (1 - c_0)^k$$

and the upper bound in the asssertion follows.

On the other hand, let $t_1 = c_1 r^{\alpha/2}$ with c_1 to be specified below. From Proposition 4.2 we get

$$P_x[\tau_{B(x,r)} < t_1] \leq c_1 c_2.$$

Observe that the constant c_2 above does not depend on c_1 provided $c_1 < 1$. Hence, we may and do choose c_1 small enough to get $c_1 c_2 < 1$. It follows that

$$E_x \tau_{B(x,r)} \geqslant t_1 P_x[\tau_{B(x,r)} > t_1] \geqslant (1 - c_1 c_2) t_1 \asymp r^{\alpha/2}$$

The proof is complete. □

5 Poisson kernel

In this section we give estimates for the Poisson kernel for balls. Since in general it follows ideas of [7], we give only a short sketch of the construction. For more detailed exposition we refer the reader to sections 5 and 6 of that article. Since the results in what follows are similar for both the homogeneous trees and hyperbolic spaces, we continue to use the notation introduced in the previous section. It should be noted, however, that this concernes the results only. The details of proofs in this section should be verified separately for each geometrical setting.

In what follows, we assume that for $x, y \in X$ the following limit exists

$$N(x,y) = \lim_{t \to 0} \frac{p_t(x,y)}{t} > 0.$$

This is verified whenever our α-stable kernel arises by a subordination of a reasonable diffusion with η_t described above. Clearly, the case of homogeneous trees and hyperbolic spaces is included. From (22) it follows that

$$N(x,y) \asymp d(x,y)^{-1-\alpha/2} V(d(x,y))^{-1}, \quad d(x,y) \geq 1. \tag{24}$$

Let

$$n(x,E) = \int_E N(x,y) d\mu(y). \tag{25}$$

For an open set D let (P_t^D) be the semigroup generated by the process killed on exiting D, i.e.

$$P_t^D f(x) = E_x[f(X_t);\ t < \tau_D].$$

This semigroup possesses transition densities denoted by $p_t^D(x,y)$ (see [9]; the argument applies here as well). Let $G_D(x,y)$ be the Green function for D, i.e. the potential for (P_t^D):

$$G_D(x,y) = \int_0^\infty p_t^D(x,y)dt.$$

With these definitions one verifies the assumptions of the Ikeda-Watanabe formula (see [19] or [7]). For homogeneous trees and hyperbolic spaces this is straightforward and we omit the details. We note, however, that at this point each geometrical structure is examined separately. We get

Proposition 5.1 (Ikeda-Watanabe formula). *Assume that $D \subset X$ is an open nonempty bounded set, $E \subset X$ is a Borel set and* $\mathrm{dist}(D,E) > 0$*. Then*

$$P_x[X_{\tau_D} \in E] = \int_D G_D(x,y)n(x,E)d\mu(y).$$

In particular, by (25) we get that $P_x[X_{\tau_D} \in \cdot]$ is absolutely continuous w.r. to μ on $(\bar{D})^c$ (this is meaningful for the hyperbolic spaces only). Let $P_D(x,\cdot)$ denote its density (i.e. Poisson kernel).

Proposition 5.2. *For any $x_0 \in X$ and $r \geq 1$ let $D = B(x_0, r)$. Then*

$$P_D(x,z) \leq c\frac{r^{\alpha/2}V(2r)}{d(x,z)^{1+\alpha/2}V(d(x,z))}, \quad z \in B(x_0,3r)^c,\ x \in D.$$

If $r \geq 2$ then

$$P_D(x,z) \geq c\frac{r^{\alpha/2}}{V(2r)d(x,z)^{1+\alpha/2}V(d(x,z))}, \quad z \in D^c,\ x \in B(x_0,r/2).$$

Proof. By (24) we have

$$P_D(x,z) \asymp \int_D \frac{G_D(x,y)}{d(y,z)^{1+\alpha/2}V(d(y,z))}d\mu(y). \tag{26}$$

Clearly, $d(y,z) \asymp d(x,z)$. Moreover, for the hyperbolic spaces and homogeneous trees we have $V(r) \asymp C_1^r$ where C_1 depends on the dimension or the degree, respectively. It follows that

$$V(d(y,z)) \geq V(d(x,z) - d(x,y)) \geq V(d(x,z) - 2r) \asymp V(2r)^{-1}V(d(x,y)).$$

Since $\int_D G_D(x,y)d\mu(y) = E_x\tau_D$ the upper bound in the assertion follows by Theorem 4.3.

On the other hand, fix $x \in B(x_0, r/2)$. Then $d(y,z) \leq cd(x,z)$, $y \in D$, $z \in D^c$. Similarly as before $V(d(y,z)) \leq V(d(y,x) + d(x,z)) \asymp V(2r)V(d(x,z))$. Moreover, $E_x\tau_D \geq E_x\tau_{B(x,r/2)} \asymp r^{\alpha/2}$. By (26) the lower bound follows. We are done. □

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